Quaid-e-Azam University
Model Paper
Mathematics A-Course
Paper A: Calculus and Analytic Geometry

Max. Marks: 100
Time Allowed: 3 hours
Note: Attempt five questions by selecting at least one and at most two
questions from each section.

Section I

Q.1. a) Show that if \( a \) is a positive real number, then \( |x| \leq a \) if and only
\[ -a \leq x \leq a. \]  
(6)
b) Solve the inequality and graph the solution set \( \frac{2}{x} - 4 > 3. \)  
(7)
c) For what value of \( a \) is
\[ f(x) = \begin{cases} 
  x^2 - 1 & x < 3 \\
  2x & x \geq 3
\end{cases} \]  
(7)
continuous at every \( x \)?

Q.2. a) Show that the function \( y = |x| \) is differentiable on \((-\infty, 0) \) and
\( (0, \infty) \) but has no derivative at \( x = 0. \)  
(6)
b) If \( y^2 + y = 2 \cos x \), find \( \frac{dy}{dx} \) at the point \( (0, 1) \).  
(7)
c) Suppose \( y = f(x) \) is continuous on a closed interval \([a, b]\) and differentiable on the open interval \((a, b)\). Then there is at least one point \( c \) in \((a, b)\) at
which
\[ \frac{f(b) - f(a)}{b - a} = f'(c). \]  
(7)

Q.3. a) Show that maximum profit (if any) occurs at a production level at
which marginal revenue equals marginal cost.  
(6)
b) If \( y = a \cos (\ln x) + b \sin (\ln x) \), prove that
\[ y^{n+2} + (2n+1)y_{n+1} + (a^2 + 1)y_n = 0. \]  
(7)
c) Find the Maclaurin's series for the function \( f(x) = \frac{1}{1-x^2}. \)  
(7)

Section II

Q.4. a) Discuss the conic represented by the equation
\[ 29x^2 - 24xy + 36y^2 + 118x - 24y - 55 = 0. \]  
(7)
b) Sketch the Lemniscate of Bernoulli
\[ r^2 = a^2 \cos 2\theta. \]  
(7)
c) Show that the equation of the tangent to the conic
\[ ax^2 + by^2 + 2kxy + 2gx + 2fy + c = 0 \]

at the point \((x_1, y_1)\) can be written in the form

\[ ax_1 + by_1 + h(x_1 + x_1y_1 + g(x_1 + y_1) + f(y_1) + c = 0. \]

Q.5. a) Find the parametric equation for the curve \(r = 2 + 3\sin \theta.\)

b) Find the angle between the straight lines

\[
\begin{align*}
\frac{x - 2}{1} &= \frac{y - 3}{1} = \frac{z + 1}{3} \\
\frac{x - 2}{2} &= \frac{y - 3}{-1} = \frac{z + 1}{3}
\end{align*}
\]

c) A straight line makes angles \(\alpha, \beta, \gamma, \delta\) with the four diagonals of a cube: prove that

\[ \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{4}{3} \]

Q.6. a) Show that the perpendicular distance of a point \((x_1, y_1, z_1)\) from the plane \(ax + by + cz + d = 0\) is \(\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}\).

b) Discuss and sketch the surface \(x^2 + 4y^2 = z^2 - 4\).

c) The cylindrical coordinates of a point are \((4, \arccos \left(\frac{1}{3}\right), 3)\) find its Cartesian coordinates and spherical coordinates.

Section III

Q.7. a) Find the height and radius of the largest right circular cylinder that can be put in a sphere of radius \(\sqrt{3}\).

b) In the cycloid \(x = a(t + \sin t), y = a(1 - \cos t)\), prove that \(\rho = 4a \cos \left(\frac{t}{2}\right)\).

c) Evaluate \(\int_0^1 e^{-x^2} dx\) by Simpson's rule with \(2m = 10\).

Q.8. a) Evaluate \(\int e^{ax} \cos (kx + c) dx\).

b) Find the reduction formula for \(\int \sin^3 x \cos^3 x dx\).

c) Find the area lying above x-axis and included between the circle \(x^2 + y^2 - 2ax = 0\) and the parabola \(y^2 = ax\).
\[ ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0 \]

at the point \((x_1, y_1)\) can be written in the form

\[ ax_1 + by_1 + h(x_1 + z; y_1) + g(x_1 + z) + f(y_1 + z) + c = 0. \]

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\begin{align*}
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